ELSEVIER
Topology and its Applications 118 (2002) 255–274
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

Moduli space of combinatorially equivalent arrangements of hyperplanes and logarithmic Gauss–Manin connections

Hiroaki Terao

Tokyo Metropolitan University, Mathematics Department, Minami-Ohsawa, Hachioji, Tokyo 192-0397, Japan

Received 30 December 1999

To the memory of Professor Nobuo Sasakura

Abstract

We consider a moduli space of combinatorially equivalent family of arrangements of hyperplanes (e.g., n distinct points in the complex line). A compactification of the moduli space is obtained by adding a boundary divisor. On the moduli space we study a Gauss–Manin connection and show that it has logarithmic poles along the boundary divisor. When the moduli space is one-codimensional, an explicit formula for the connection matrix is given.

AMS classification: 52B30; 33C70

Keywords: Arrangements of hyperplanes; Gauss–Manin connections; Local system cohomology; Combinatorial equivalence; β**nbc** basis

1. Introduction

Fix a pair (ℓ, n) with $\ell \geqslant 1$ and $n \geqslant 0$. Let $\mathcal{A}_n(\mathbb{C}^\ell)$ be the set of affine arrangements of n distinct linearly ordered hyperplanes in $\mathbb{C}^\ell$. In other words, each element $\mathcal{A}$ of $\mathcal{A}_n(\mathbb{C}^\ell)$ is a collection $\{H_1, \ldots, H_n\}$ where $H_1, \ldots, H_n$ are distinct affine hyperplanes in $\mathbb{C}^\ell$. Two arrangements $\mathcal{A}^{(i)} := \{H_1^{(i)}, \ldots, H_n^{(i)}\} \in \mathcal{A}_n(\mathbb{C}^\ell)$ $(i = 1, 2)$ are said to be *combinatorially equivalent*, denoted by $\mathcal{A}^{(1)} \sim \mathcal{A}^{(2)}$, if

$$\dim H_{i_1}^{(1)} \cap \cdots \cap H_{i_p}^{(1)} = \dim H_{i_1}^{(2)} \cap \cdots \cap H_{i_p}^{(2)}$$

for each $(i_1, \ldots, i_p)$, $1 \leqslant i_1 < \cdots < i_p \leqslant n$. Here we agree that the dimension of the empty set is equal to -1.

E-mail address: hterao@comp.metro-u.ac.jp (H. Terao).

0166-8641/02/$ – see front matter
PII: S0166-8641(01)00054-2

Fix $\mathcal{A} := \{H_1, \ldots, H_n\} \in \mathcal{A}_n(\mathbb{C}^\ell)$. Let

$$\mathsf{B}_{\mathcal{A}} = \mathsf{B} := \{\mathcal{B} \in \mathcal{A}_n(\mathbb{C}^\ell) \mid \mathcal{B} \sim \mathcal{A}\}.$$

In Section 2, we naturally identify B with a locally closed subset of $((\mathbb{CP}^\ell)^*)^n$, where $(\mathbb{CP}^\ell)^*$ is the ℓ-dimensional dual complex projective space. Let $\overline{\mathsf{B}}$ be the closure of B in $((\mathbb{CP}^\ell)^*)^n$. Then, as we will see in Proposition 4, the boundary $\mathsf{D} := \overline{\mathsf{B}} \setminus \mathsf{B}$ is defined by a single equation on $\overline{\mathsf{B}}$. The hypersurface D, in general, has several irreducible components. When $\mathcal{A}$ is of general position (i.e., $\dim H_{i_1} \cap \cdots \cap H_{i_p} = \ell - p$ if $1 \leqslant i_1 < \cdots < i_p \leqslant n$ and $1 \leqslant p \leqslant \ell + 1$), B is dense in $((\mathbb{CP}^\ell)^*)^n$: $\overline{\mathsf{B}} = ((\mathbb{CP}^\ell)^*)^n$. In this case D has $\binom{n+1}{\ell+1}$ irreducible components. We can also describe the geometry of D when the codimension of B in $((\mathbb{CP}^\ell)^*)^n$ is equal to one. In this case, D has $\binom{n+1}{\ell+1} - \ell(n - \ell - 1)$ irreducible components (if $\ell \geqslant 2$) or $n(n-1)/2$ irreducible components (if $\ell = 1$). The study of D is naturally related to the theory of determinantal ideals.

We assume that $\mathcal{A}$ is *essential*, i.e., there exist ℓ hyperplanes in $\mathcal{A}$ whose intersection is a point. In particular, $n = |\mathcal{A}| \geqslant \ell$. We have a topological fibration

$$\pi : \mathsf{M} \to \mathsf{B}$$

such that $\pi^{-1}(\boldsymbol{t}) = \mathsf{M}_{\boldsymbol{t}} = M(\mathcal{A}_{\boldsymbol{t}}) := \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}_{\boldsymbol{t}}} H$ for each $\boldsymbol{t} \in \mathsf{B}$. Here $\mathcal{A}_{\boldsymbol{t}} \in \mathcal{A}_n(\mathbb{C}^\ell)$ is the arrangement corresponding to $\boldsymbol{t} \in \mathsf{B}$. Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$. We call λ a *weight*. Then λ defines a rank-one local system $\mathcal{L}_\lambda$ on $\mathsf{M}_{\boldsymbol{t}}$ so that $\mathcal{L}_\lambda$ has monodromy $\exp(-2\pi\sqrt{-1}\lambda_i)$ around $H_i \in \mathcal{A}_{\boldsymbol{t}}$ $(1 \leqslant i \leqslant n)$. Let $\mathcal{L}_\lambda^\vee$ be the dual local system which has monodromy $\exp(2\pi\sqrt{-1}\lambda_i)$ around H_i. Then, for a sufficiently generic λ, there exists a local system $\mathcal{H}_\ell \to \mathsf{B}$ of rank β whose fiber at $\boldsymbol{t} \in \mathsf{B}$ is equal to the local system homology $H_\ell(\mathsf{M}_{\boldsymbol{t}}, \mathcal{L}_\lambda^\vee)$ because π is locally trivial. Here β is the absolute value of the Euler characteristic of $\mathsf{M}_{\boldsymbol{t}}$, which is independent of choice of $\boldsymbol{t} \in \mathsf{B}$. We can also define, in the dual manner, a local system $\mathcal{H}^\ell \to \mathsf{B}$ of rank β whose fiber at $\boldsymbol{t} \in \mathsf{B}$ is equal to the local system cohomology $H^\ell(\mathsf{M}_{\boldsymbol{t}}, \mathcal{L}_\lambda)$ so that $\mathcal{H}^\ell$ is a globally trivial local system with the β**nbc** global frame $\Xi_1, \ldots, \Xi_\beta \in \Gamma(\mathsf{B}, \mathcal{H}^\ell)$ [3, 3.9]. Let $\alpha_i = 0$ be a defining equation of $H_i \in \mathcal{A}_{\boldsymbol{t}}$ and let $\Phi_\lambda = \alpha_1^{\lambda_1} \ldots \alpha_n^{\lambda_n}$. Then Φ_λ is a multi-valued holomorphic function on $\mathsf{M}_{\boldsymbol{t}}$ and gives a section of $\mathcal{L}_\lambda^\vee$. The (hypergeometric) pairing

$$\langle\,,\rangle : \mathcal{H}_\ell \times \mathcal{H}^\ell \to \mathcal{O}_{\mathsf{B}}$$

is defined by the (hypergeometric) integral (in the sense of Aomoto–Gelfand)

$$\langle \sigma, \omega \rangle = \int_\sigma \Phi_\lambda \omega,$$

where $\mathcal{O}_{\mathsf{B}}$ is the sheaf of germs of holomorphic functions on (the smooth part of) B. Let σ be a flat local section on an open set $U \subseteq \mathsf{B}$ of the local system $\mathcal{H}_\ell$. Then the section $\sigma : U \to \mathcal{H}_\ell$ is represented by the vector

$$\widetilde{\sigma}(\boldsymbol{t}) := \begin{pmatrix} \langle \sigma(\boldsymbol{t}), \Xi_1(\boldsymbol{t}) \rangle \\ \cdots \\ \cdots \\ \langle \sigma(\boldsymbol{t}), \Xi_\beta(\boldsymbol{t}) \rangle \end{pmatrix}$$

for each $\boldsymbol{t} \in U$. Thus the vector $\widetilde{\sigma}$, which is a coordinate vector for the flat section σ, satisfies a system

$$d'\widetilde{\sigma} = \Omega \wedge \widetilde{\sigma},$$

of differential equations of the first order, where d' is the exterior differential on B and Ω is a $\beta \times \beta$-matrix whose entry is a differential 1-form on (the smooth part of) B. This matrix Ω is a Gauss–Manin connection matrix satisfying $d'\Omega - \Omega \wedge \Omega = 0$. In Section 3, we will show that each entry of the connection matrix Ω has at most logarithmic poles along $\mathsf{D} = \overline{\mathsf{B}} \setminus \mathsf{B}$. Since the geometry of D is sufficiently understood when the codimension of B in $((\mathbb{CP}^\ell)^*)^n$ is one, we have an explicit formula for Ω in this case in Section 4. It turns out that each entry of Ω is a linear combination of logarithmic forms with poles along each irreducible component of D with coefficients in $\sum_{i=1}^n \mathbb{Z}\lambda_i$. It might be natural to ask if it is in the case for any B of higher codimensions.

2. Combinatorially equivalent family of arrangements

We compactify $\mathbb{C}^\ell$ by adding the infinite hyperplane $\overline{H}_\infty$ to get complex projective space $\mathbb{CP}^\ell$.

Definition 1. A multiset is a set which allows repetitions. A multiset $\mathcal{M}$ is a projective *multiarrangement* if $\mathcal{M}$ is a finite multiset of projective hyperplanes of $\mathbb{CP}^\ell$. Let

$$\mathcal{M}_n\bigl(\mathbb{CP}^\ell\bigr) = \bigl\{\text{projective multiarrangements of } n+1 \text{ linearly ordered hyperplanes of } \mathbb{CP}^\ell \text{ where } \overline{H}_\infty \text{ is the last hyperplane}\bigr\}.$$

Each point of $(\mathbb{CP}^\ell)^*$ corresponds to a hyperplane of $\mathbb{CP}^\ell$. Thus we identify $\mathcal{M}_n(\mathbb{CP}^\ell)$ with $((\mathbb{CP}^\ell)^*)^n$:

$$\mathcal{M}_n\bigl(\mathbb{CP}^\ell\bigr) = \bigl(\bigl(\mathbb{CP}^\ell\bigr)^*\bigr)^n.$$

Then $\mathcal{M}_n(\mathbb{CP}^\ell)$ is a compact complex manifold biholomorphic to $(\mathbb{CP}^\ell)^n$.

Let $\mathcal{M} \in \mathcal{M}_n(\mathbb{CP}^\ell)$. Write $\mathcal{M} = \{\overline{H}_1, \overline{H}_2, \ldots, \overline{H}_{n+1} = \overline{H}_\infty\}$. We say that $\mathcal{M}$ is *essential* if $\bigcap_{H\in\mathcal{M}} H = \emptyset$. Denote the set $\{1, 2, \ldots, n+1\}$ by $[n+1]$. Define

$$\left(\binom{[n+1]}{\ell+1}\right) = \bigl\{\text{subsets of } [n+1] \text{ of cardinality } \ell+1\bigr\}.$$

Let $\wp$ denote the power set. Let

$$\mathcal{J} : \mathcal{M}_n\bigl(\mathbb{CP}^\ell\bigr) \to \wp\left(\binom{[n+1]}{\ell+1}\right)$$

be the map defined by

$$\begin{aligned}\mathcal{J}(\mathcal{M}) &= \left\{\{i_1, \ldots, i_{\ell+1}\} \in \left(\binom{[n+1]}{\ell+1}\right) \,\middle|\, \overline{H}_{i_1} \cap \cdots \cap \overline{H}_{i_{\ell+1}} \neq \emptyset\right\}\\ &= \left\{\{i_1, \ldots, i_{\ell+1}\} \in \left(\binom{[n+1]}{\ell+1}\right) \,\middle|\, \bigl\{\overline{H}_{i_1}, \ldots, \overline{H}_{i_{\ell+1}}\bigr\} \text{ is not essential}\right\}.\end{aligned}$$

Recall that $\mathcal{A}_n(\mathbb{C}^\ell)$ is the set of affine arrangements of n linearly ordered distinct hyperplanes in $\mathbb{C}^\ell$. When we want to emphasize that repetitions are *not* allowed, we call an arrangement *simple*. Let $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$. The projective closure $\mathcal{A}_\infty$ of $\mathcal{A}$ is defined by

$$\mathcal{A}_\infty = \{\overline{H} \mid H \in \mathcal{A}\} \cup \{\overline{H}_\infty\},$$

where $\overline{H}$ is the closure of H in $\mathbb{CP}^\ell$. The hyperplanes of $\mathcal{A}_\infty$ are naturally linearly ordered by regarding the infinite hyperplane $\overline{H}_\infty$ as the last, or the $(n+1)$st hyperplane. Thus $\mathcal{A}_\infty \in \mathcal{M}_n(\mathbb{CP}^\ell)$ and there is an injective map

$$\mathcal{A}_n(\mathbb{C}^\ell) \to \mathcal{M}_n(\mathbb{CP}^\ell)$$

which sends $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$ to its projective closure $\mathcal{A}_\infty \in \mathcal{M}_n(\mathbb{CP}^\ell)$. Through this injection, we identify $\mathcal{A}_n(\mathbb{C}^\ell)$ with its image in $\mathcal{M}_n(\mathbb{CP}^\ell)$. Then the subset $\mathcal{A}_n(\mathbb{C}^\ell)$ is open dense in $\mathcal{M}_n(\mathbb{CP}^\ell) \simeq ((\mathbb{CP}^\ell)^*)^n$ with respect to the Zariski topology because it is characterized by the open condition that no two hyperplanes are equal.

Proposition 2. *Let $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$. Then the following three conditions are equivalent*:

(i) *$\mathcal{A}$ is essential, i.e., there exist ℓ hyperplanes in $\mathcal{A}$ whose intersection is a point,*
(ii) *$\mathcal{A}_\infty$ is essential, i.e., the intersection of all hyperplanes of $\mathcal{A}_\infty$ is empty,*
(iii) *$\mathcal{J}(\mathcal{A}_\infty) \neq \left(\binom{[n+1]}{\ell+1}\right)$.*

Proof. It is clear that conditions (ii) and (iii) are equivalent because $\bigcap_{\overline{H} \in \mathcal{A}_\infty} \overline{H} = \emptyset$ implies that there exist $\ell + 1$ hyperplanes $\overline{H}_{i_1}, \ldots, \overline{H}_{i_{\ell+1}} \in \mathcal{A}_\infty$ whose intersection is empty.

We also have: (iii) $\Leftrightarrow$ there exist $\ell + 1$ hyperplanes $\overline{H}_{i_1}, \ldots, \overline{H}_{i_{\ell+1}} \in \mathcal{A}_\infty$ such that $\overline{H}_{i_1} \cap \cdots \cap \overline{H}_{i_{\ell+1}} = \emptyset \Leftrightarrow$ there exist ℓ hyperplanes $H_{i_1}, \ldots, H_{i_\ell} \in \mathcal{A}$ such that $\overline{H}_{i_1} \cap \cdots \cap \overline{H}_{i_\ell} \cap \overline{H}_\infty = \emptyset \Leftrightarrow$ there exist ℓ hyperplanes $H_{i_1}, \ldots, H_{i_\ell} \in \mathcal{A}$ such that $H_{i_1} \cap \cdots \cap H_{i_\ell}$ is a point $\Leftrightarrow$ (i). □

Proposition 3. *Let $\mathcal{A}^{(1)}, \mathcal{A}^{(2)} \in \mathcal{A}_n(\mathbb{C}^\ell)$ be essential simple arrangements with an order-preserving bijection $\iota : \mathcal{A}^{(1)} \to \mathcal{A}^{(2)}$. Then the following two conditions are equivalent*:

(i) *$\mathcal{A}^{(1)} \sim \mathcal{A}^{(2)}$, i.e., $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ are combinatorially equivalent,*
(ii) *$\mathcal{J}(\mathcal{A}_\infty^{(1)}) = \mathcal{J}(\mathcal{A}_\infty^{(2)})$.*

Proof. Let $\mathcal{A}^{(k)} = \{H_1^{(k)}, \ldots, H_n^{(k)}\}$ $(k = 1, 2)$.

(i) $\Rightarrow$ (ii): Suppose $1 \leqslant i_1 < \cdots < i_{\ell+1} \leqslant n+1$.

Case 1. If $i_{\ell+1} = n + 1$, then, for $k = 1, 2$, we have:

$$\begin{aligned}\{i_1, \ldots, i_{\ell+1}\} \notin \mathcal{J}(\mathcal{A}_\infty^{(k)}) &\Leftrightarrow \overline{H}_{i_1}^{(k)} \cap \cdots \cap \overline{H}_{i_\ell}^{(k)} \cap \overline{H}_\infty = \emptyset \\ &\Leftrightarrow H_{i_1}^{(k)} \cap \cdots \cap H_{i_\ell}^{(k)} \text{ is a point} \\ &\Leftrightarrow \dim(H_{i_1}^{(k)} \cap \cdots \cap H_{i_\ell}^{(k)}) = 0.\end{aligned}$$

Therefore $\{i_1, \ldots, i_{\ell+1}\} \in \mathcal{J}(\mathcal{A}_\infty^{(1)})$ if and only if $\{i_1, \ldots, i_{\ell+1}\} \in \mathcal{J}(\mathcal{A}_\infty^{(2)})$.

Case 2. If $i_{\ell+1} < n + 1$, then let $\mathcal{B}^{(k)} = \{H_{i_1}^{(k)}, \ldots, H_{i_{\ell+1}}^{(k)}\}$ for $k = 1, 2$. Then we have:

$$
\begin{aligned}
&\{i_1, \dots, i_{\ell+1}\} \notin \mathcal{J}\big(\mathcal{A}_\infty^{(k)}\big)\\
&\quad\Leftrightarrow \overline{H}_{i_1}^{(k)} \cap \dots \cap \overline{H}_{i_{\ell+1}}^{(k)} = \emptyset\\
&\quad\Leftrightarrow H_{i_1}^{(k)} \cap \dots \cap H_{i_{\ell+1}}^{(k)} = \emptyset \text{ and } \overline{H}_{i_1}^{(k)} \cap \dots \cap \overline{H}_{i_{\ell+1}}^{(k)} \cap \overline{H}_\infty = \emptyset\\
&\quad\Leftrightarrow H_{i_1}^{(k)} \cap \dots \cap H_{i_{\ell+1}}^{(k)} = \emptyset \text{ and } \mathcal{B}_\infty^{(k)} \text{ is essential}\\
&\quad\Leftrightarrow H_{i_1}^{(k)} \cap \dots \cap H_{i_{\ell+1}}^{(k)} = \emptyset \text{ and } \mathcal{B}^{(k)} \text{ is essential}\\
&\quad\Leftrightarrow \dim\big(H_{i_1}^{(k)} \cap \dots \cap H_{i_{\ell+1}}^{(k)}\big) = -1 \text{ and there exist } \ell \text{ hyperplanes in } \mathcal{B}^{(k)}\\
&\qquad \text{whose intersection is zero-dimensional.}
\end{aligned}
$$

Here the penultimate equivalence follows from Proposition 2. Therefore $\{i_1, \dots, i_{\ell+1}\} \in \mathcal{J}(\mathcal{A}_\infty^{(1)})$ if and only if $\{i_1, \dots, i_{\ell+1}\} \in \mathcal{J}(\mathcal{A}_\infty^{(2)})$.

(ii) $\Rightarrow$ (i): Let $1 \leqslant i_1 < \dots < i_p \leqslant n$. Since $\mathcal{A}$ is essential, we have

$$
\begin{aligned}
&\dim H_{i_1}^{(k)} \cap \dots \cap H_{i_p}^{(k)} = p\\
&\quad\Leftrightarrow \text{there exist } 1 \leqslant i_{p+1} < \ldots < i_\ell \leqslant n \text{ such that } \dim H_{i_1}^{(k)} \cap \dots \cap H_{i_\ell}^{(k)} = 0\\
&\quad\Leftrightarrow \text{there exist } 1 \leqslant i_{p+1} < \dots < i_\ell \leqslant n \text{ such that } \overline{H}_{i_1}^{(k)} \cap \dots \cap \overline{H}_{i_\ell}^{(k)} \cap \overline{H}_\infty = \emptyset\\
&\quad\Leftrightarrow \text{there exist } 1 \leqslant i_{p+1} < \dots < i_\ell \leqslant n\\
&\qquad \text{such that } \{i_1, \dots, i_\ell, n+1\} \in \mathcal{J}\big(\mathcal{A}_\infty^{(1)}\big) = \mathcal{J}\big(\mathcal{A}_\infty^{(2)}\big)
\end{aligned}
$$

for $k = 1, 2$. Therefore $\dim H_{i_1}^{(1)} \cap \dots \cap H_{i_p}^{(1)} = p$ if and only if $\dim H_{i_1}^{(2)} \cap \dots \cap H_{i_p}^{(2)} = p$. The condition (i) easily follows from this. □

Let $(u_0 : \dots : u_\ell)$ be the homogeneous coordinates for $\mathbb{CP}^\ell = \mathbb{C}^\ell \cup \overline{H}_\infty$ so that the equation $u_0 = 0$ defines $\overline{H}_\infty$. Let $\boldsymbol{t}$ denote the ordered $(n+1)$-tuple of homogeneous coordinates:

$$
\boldsymbol{t} = \big(\big(t_1^{(0)} : \dots : t_1^{(\ell)}\big), \big(t_2^{(0)} : \dots : t_2^{(\ell)}\big), \dots, \big(t_n^{(0)} : \dots : t_n^{(\ell)}\big)\big).
$$

Use $\boldsymbol{t}$ as the homogeneous coordinates for $((\mathbb{CP}^\ell)^*)^n$. The point $\boldsymbol{t}$ of $((\mathbb{CP}^\ell)^*)^n$ corresponds to the projective multiarrangement $\mathcal{M}_{\boldsymbol{t}}$ whose hyperplanes are $\overline{H}_i$ defined by $\overline{\alpha}_i := \sum_{j=0}^{\ell} t_i^{(j)} u_j = 0$ $(i = 1, \dots, n)$ and $\overline{H}_{n+1} = \overline{H}_\infty$ defined by $\overline{\alpha}_{n+1} := u_0 = 0$. Define the $(\ell+1) \times (n+1)$-matrix T by

$$
\mathsf{T} = \begin{pmatrix} t_1^{(0)} & \cdots & t_n^{(0)} & 1\\ t_1^{(1)} & \cdots & t_n^{(1)} & 0\\ \vdots & \ddots & \vdots & \vdots\\ t_1^{(\ell)} & \cdots & t_n^{(\ell)} & 0 \end{pmatrix}.
$$

Note that the ith column of T gives the coefficients of $\overline{\alpha}_i$ $(i = 1, \ldots, n+1)$. Let $S \in \left(\binom{[n+1]}{\ell+1}\right)$. Denote by Δ_S the $(\ell+1)$-minor using the columns of T corresponding to S. Then it is easy to see

$$\mathcal{J}(\mathcal{M}_{\boldsymbol{t}}) = \left\{ S \in \left(\binom{[n+1]}{\ell+1}\right) \;\middle|\; \Delta_S(\boldsymbol{t}) = 0 \right\}$$

by definition. Let $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$. Define

$$\mathsf{B}_{\mathcal{A}} := \mathcal{J}^{-1}\bigl(\mathcal{J}(\mathcal{A}_\infty)\bigr).$$

Let $\mathcal{J} \subseteq \left(\binom{[n+1]}{\ell+1}\right)$. Define

$$\mathsf{C}_{\mathcal{J}} = \left\{ \boldsymbol{t} \in \left(\left(\mathbb{CP}^\ell\right)^*\right)^n \mid \Delta_S(\boldsymbol{t}) = 0 \text{ for } S \in \mathcal{J} \right\}.$$

Then $\mathsf{C}_{\mathcal{J}}$ is Zariski closed in $((\mathbb{CP}^\ell)^*)^n$ and thus compact.

Proposition 4. *Let $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$ be essential. Write $\mathsf{B} = \mathsf{B}_{\mathcal{A}}$. Then*

(i) $\mathsf{B} = \{\mathcal{B} \in \mathcal{A}_n(\mathbb{C}^\ell) \mid \mathcal{A} \sim \mathcal{B}\}$,

(ii) $\mathsf{B} = \{\boldsymbol{t} \in ((\mathbb{CP}^\ell)^*)^n \mid \Delta_S(\boldsymbol{t})$ *vanishes exactly when* $S \in \mathcal{J}(\mathcal{A}_\infty)\}$,

(iii) B *is a locally closed subset of* $((\mathbb{CP}^\ell)^*)^n$,

(iv) *Let $\overline{\mathsf{B}}$ be the closure of B in $((\mathbb{CP}^\ell)^*)^n$. Define $\mathsf{D}_T = \overline{\mathsf{B}} \cap \mathsf{C}_{\{T\}}$ for*

$$T \in \mathcal{J}(\mathcal{A}_\infty)^c := \left(\binom{[n+1]}{\ell+1}\right) \setminus \mathcal{J}(\mathcal{A}_\infty).$$

Then

$$\mathsf{D} := \overline{\mathsf{B}} \setminus \mathsf{B} = \bigcup_{T \in \mathcal{J}(\mathcal{A}_\infty)^c} \mathsf{D}_T$$

and each D_T is a hypersurface in $\overline{\mathsf{B}}$.

Proof. (i) Suppose $\mathcal{A} = \{H_1, \ldots, H_n\}$ and $\mathcal{M} = \{\overline{K}_1, \ldots, \overline{K}_{n+1}\} \in \mathsf{B}$. Then $\mathcal{J}(\mathcal{M}) = \mathcal{J}(\mathcal{A}_\infty)$. We will first show that $\mathcal{M} = \mathcal{B}_\infty$ for some $\mathcal{B} \in \mathcal{A}_n(\mathbb{C}^\ell)$. If not, $\mathcal{M}$ has a hyperplane of multiplicity more than one. Suppose $\overline{K_i} = \overline{K_j}$ $(i \neq j)$. Then $S \in \mathcal{J}(\mathcal{M}) = \mathcal{J}(\mathcal{A}_\infty)$ whenever S contains i and j. Since $\mathcal{A}$ is essential, this implies $\overline{H_i} = \overline{H_j} \in \mathcal{A}_\infty$ $(i \neq j)$, which contradicts the fact that $\mathcal{A}$ is simple. Thus there exists $\mathcal{B} \in \mathcal{A}_n(\mathbb{C}^\ell)$ with $\mathcal{M} = \mathcal{B}_\infty$. Since $\mathcal{A}$ is essential, so is $\mathcal{B}$ by Proposition 2. Then apply Proposition 3.

(ii) One has

$$\begin{aligned} \mathcal{M}_{\boldsymbol{t}} \in \mathsf{B} &\Leftrightarrow \mathcal{J}(\mathcal{A}_\infty) = \mathcal{J}(\mathcal{M}_{\boldsymbol{t}}) \\ &\Leftrightarrow \mathcal{J}(\mathcal{A}_\infty) = \left\{ S \in \left(\binom{[n+1]}{\ell+1}\right) \;\middle|\; \Delta_S(\boldsymbol{t}) = 0 \right\}. \end{aligned}$$

(iii) By (ii), one has

$$\mathsf{B} = \mathsf{C}_{\mathcal{J}(\mathcal{A}_\infty)} \setminus \bigcup_{T \in \mathcal{J}(\mathcal{A}_\infty)^c} \mathsf{C}_{\{T\}}.$$

Thus B is locally closed.

(iv) One has

$$\mathsf{D} = \overline{\mathsf{B}} \setminus \mathsf{B} = \overline{\mathsf{B}} \cap \left(\bigcup_{T \in \mathcal{J}(\mathcal{A}_\infty)^c} \mathsf{C}_{\{T\}} \right) = \bigcup_{T \in \mathcal{J}(\mathcal{A}_\infty)^c} \mathsf{D}_T.$$

Note that D_T $(T \in \mathcal{J}(\mathcal{A}_\infty)^c)$ is defined by a single equation in $\overline{\mathsf{B}}$. If D_T is not of codimension one in $\overline{\mathsf{B}}$, then there exists an irreducible component C_0 of $\overline{\mathsf{B}}$ which lies in D_T. Thus $C_0 \cap \mathsf{B} = \emptyset$. On the other hand, since B is dense in $\overline{\mathsf{B}}$, B meets any irreducible component of $\overline{\mathsf{B}}$. This is a contradiction, which proves (iv). $\square$

Let $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$ be essential in the rest of this section. In particular, $\ell \leqslant n$. Because of Proposition 4(i), we can regard $\mathsf{B}_\mathcal{A}$ as a moduli space of the affine arrangements which are combinatorially equivalent to $\mathcal{A}$. When the codimension of $\mathsf{B}_\mathcal{A}$ in $((\mathbb{CP}^\ell)^*)^n$ is less than two, we can explicitly describe the geometry of $\mathsf{B}_\mathcal{A}$ and $\mathsf{D}_\mathcal{A}$.

Codimension-zero case: The moduli space $\mathsf{B}_\mathcal{A}$ is zero-codimensional in $((\mathbb{CP}^\ell)^*)^n$ if and only if $|\mathcal{J}(\mathcal{A}_\infty)| = 0$. We say that an affine arrangement $\mathcal{A}$ is of general position if $\mathcal{J}(\mathcal{A}_\infty) = \emptyset$. Thus $\mathsf{B}_\mathcal{A}$ is a moduli space of affine arrangements of general position. In this case $\mathsf{B}_\mathcal{A}$ is a dense open subset of $((\mathbb{CP}^\ell)^*)^n$ and

$$\mathsf{D}_\mathcal{A} = \bigcup_T \mathsf{C}_{\{T\}},$$

where T runs over $\left(\binom{[n+1]}{\ell+1}\right)$. Since $\mathsf{C}_{\{T\}}$ is defined by the single equation $\Delta_T = 0$ and the determinant function is an irreducible polynomial (a special case of Theorem 6), each $\mathsf{C}_{\{T\}}$ is an irreducible hypersurface. Therefore $\mathsf{D}_\mathcal{A}$ is composed of $\binom{n+1}{\ell+1}$ irreducible components. When $\ell = 1$, $\mathsf{B}_\mathcal{A} = \{(t_1, \ldots, t_n) \in \mathbb{C}^n \mid t_i \neq t_j \ (i \neq j)\}$ is the pure braid space.

Codimension-one case: The moduli space $\mathsf{B}_\mathcal{A}$ is one-codimensional in $((\mathbb{CP}^\ell)^*)^n$ if and only if $|\mathcal{J}(\mathcal{A}_\infty)| = 1$.

Proposition 5. *Suppose $\mathcal{J}(\mathcal{A}_\infty) = \{S\}$. Write $\mathsf{B} = \mathsf{B}_\mathcal{A}$, $\mathsf{C} = \mathsf{C}_{\{S\}}$ and $\mathsf{D} = \overline{\mathsf{B}} \setminus \mathsf{B}$. Then*

(i) $\overline{\mathsf{B}} = \mathsf{C}$ *is irreducible,*

(ii) B *is smooth,*

(iii) *the irreducible components of* D *are*:

Type I. $\mathsf{C}_{\{S,S'\}}$ *for* $S' \in \left(\binom{[n+1]}{\ell+1}\right)$ *with* $|S \cap S'| \leqslant \ell - 1$,

Type II. $\mathsf{C}_{\langle S-p\rangle}$ *for* $p \in S$, *where* $\langle S - p\rangle := \{S' \in \left(\binom{[n+1]}{\ell+1}\right) \mid S' \supseteq S \setminus \{p\}\}$, *and*

Type III. $\mathsf{C}_{\langle S+q\rangle}$ *for* $q \in [n+1] \setminus S$, *where* $\langle S + q\rangle := \{S' \in \left(\binom{[n+1]}{\ell+1}\right) \mid S' \subseteq S \cup \{q\}\}$.

In all, there exist $\binom{n+1}{\ell+1} - \ell(n - \ell - 1)$ *irreducible components of* D. *When* $\ell = 1$, *the Type II does not appear and the number of irreducible components of* D *is equal to* $n(n-1)/2$.

In order to prove this proposition we need the following fundamental result on determinantal ideals:

Theorem 6 (Hochter–Eagon [4]). *Let $X = (X_{ij})$ be a matrix of indeterminates over an integral domain R of size $m \times n$. Let $I_t(X)$ be the ideal in the polynomial ring $R[X_{ij}]$ generated by the t-minors of X. Then $I_t(X)$ is a prime ideal of height $(m-t+1)(n-t+1)$.*

Recall the $(\ell+1) \times (n+1)$-matrix T:

$$\mathsf{T} = \begin{pmatrix} t_1^{(0)} & \cdots & t_n^{(0)} & 1 \\ t_1^{(1)} & \cdots & t_n^{(1)} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ t_1^{(\ell)} & \cdots & t_n^{(\ell)} & 0 \end{pmatrix}.$$

Let $\mathbb{C}[\mathsf{T}]$ be the polynomial ring over $\mathbb{C}$ with indeterminates $\{t_j^{(i)}\}_{0 \leqslant i \leqslant \ell,\, 1 \leqslant j \leqslant n}$. For $S \subseteq [n+1]$, define T_S to be the submatrix of T consisting of the columns corresponding to S. When $|S| = \ell+1$, $\Delta_S = \det(\mathsf{T}_S)$.

Lemma 7. *Let $S, S' \in \left(\binom{[n+1]}{\ell+1}\right)$ and $I := (\Delta_S, \Delta_{S'})\mathbb{C}[\mathsf{T}]$. Then*
(i) $I = I_\ell(\mathsf{T}_{S \cap S'}) \cap I_{\ell+1}(\mathsf{T}_{S \cup S'})$
when $|S \cap S'| = \ell$. Here I_t is defined in the same manner as in Theorem 6.
(ii) *I is a prime ideal of height two when $|S \cap S'| \leqslant \ell - 1$.*

Proof. (i) Let $A = \mathsf{T}_{S \cap S'}$, $B = \mathsf{T}_{S \cup S'}$. Write $B = (b_{ij})_{0 \leqslant i \leqslant \ell,\, 0 \leqslant j \leqslant \ell+1}$. Define

$$\Delta_j = (-1)^j \det(B_j) \quad (j = 0, \ldots, \ell+1),$$

where B_j is obtained from B by deleting the jth column of B. We may assume that $\Delta_S = \Delta_0$ and $\Delta_{S'} = \Delta_{\ell+1}$. Let $P_1 := I_\ell(A)$ and $P_2 := I_{\ell+1}(B) = (\Delta_0, \ldots, \Delta_{\ell+1})$. We will show $I = P_1 \cap P_2$. If $\ell = 1$ and $S \cap S' = \{n+1\}$, then $P_1 = \mathbb{C}[\mathsf{T}]$. In this case $I = P_2$ and (i) holds true. In the other cases, both P_1 and P_2 are prime ideals of height two by Theorem 6. By elementary linear algebra, one has

$$\sum_{j=0}^{\ell+1} b_{ij} \Delta_j = 0 \quad (i = 0, \ldots, \ell).$$

Thus

$$\sum_{j=1}^{\ell} b_{ij} \Delta_j \in I \quad (i = 0, \ldots, \ell).$$

By applying Cramer's rule, one obtains $P_1 P_2 \subseteq I$. Since I is generated by two irreducible polynomials, every associated prime of I is of height two or less. If P is an associated prime of I, then $P_1P_2 \subseteq I \subseteq P$. Thus either $P_1 \subseteq P$ or $P_2 \subseteq P$. So we have $P \in \{P_1, P_2\}$. Write a primary decomposition of I as

$$I = Q_1 \cap Q_2$$

with $\sqrt{Q_i} = P_i$ $(i = 1, 2)$. Note that there is no inclusion relation between P_1 and P_2. Since $P_1P_2 \subseteq Q_i$, we have $P_i = Q_i$ $(i = 1, 2)$.

(ii) (K. Kurano) *Case* 1: Suppose $n + 1 \notin S \cap S'$. Choose $S'' \in \left(\binom{[n+1]}{\ell+1}\right)$ such that $S \cap S' \subset S'' \subset S \cup S'$ and $|S \cap S''| = \ell$. Let $\Delta = \Delta_S$, $\Delta' = \Delta_{S'}$, and $\Delta'' = \Delta_{S''}$. By abuse of notation, let a matrix also denote the set of its entries. So the ring $R := \mathbb{C}\left[\mathsf{T}_{S''}, (\Delta'')^{-1}\right]$ stands for the subring of $\mathbb{C}(\mathsf{T}_{S''})$ generated by $(\Delta'')^{-1}$ and the entries of $\mathsf{T}_{S''}$ over $\mathbb{C}$. Let $\mathsf{Z} := (\mathsf{T}_{S''})^{-1}$. Then each entry of Z lies in R. Let $S''' := (S \cup S') \setminus S''$. Since the entries of $\mathsf{T}_{S'''}$ are algebraically independent over $\mathbb{C}(\mathsf{T}_{S''})$, so are the entries of $\mathsf{Z}\mathsf{T}_{S'''}$. Note that there exists an entry of $\mathsf{Z}\mathsf{T}_{S'''}$ which is equal either to $\det(\mathsf{Z}\mathsf{T}_S)$ or to $-\det(\mathsf{Z}\mathsf{T}_S)$ and that there exists a minor of $\mathsf{Z}\mathsf{T}_{S'''}$ which is equal either to $\det(\mathsf{Z}\mathsf{T}_{S'})$ or to $-\det(\mathsf{Z}\mathsf{T}_{S'})$. Thus the ideal

$$\begin{aligned}\left(\Delta, \Delta'\right)R\left[\mathsf{T}_{S'''}\right] &= \left(\det(\mathsf{T}_S), \det(\mathsf{T}_{S'})\right)R\left[\mathsf{T}_{S'''}\right] \\ &= \left(\det(\mathsf{Z}\mathsf{T}_S), \det(\mathsf{Z}\mathsf{T}_{S'})\right)R\left[\mathsf{Z}\mathsf{T}_{S'''}\right]\end{aligned}$$

is a prime ideal of

$$R\left[\mathsf{Z}\mathsf{T}_{S'''}\right] = R\left[\mathsf{T}_{S'''}\right] = \mathbb{C}\left[\mathsf{T}_{S\cup S'}, (\Delta'')^{-1}\right]$$

by Theorem 6. On the other hand, the associated primes of $(\Delta, \Delta')R\left[\mathsf{T}_{S\cup S'}\right]$ are $I_\ell(\mathsf{T}_{S\cap S''})$ and $I_{\ell+1}(\mathsf{T}_{S\cup S''})$. Since $(S \cap S'') \setminus S' \neq \emptyset$ and $|(S \cup S'') \setminus S'| \geqslant 2$, one has $\Delta' \notin I_\ell(\mathsf{T}_{S\cap S''})$ and $\Delta' \notin I_{\ell+1}(\mathsf{T}_{S\cup S''})$. Therefore $(\Delta, \Delta'')\colon (\Delta') = (\Delta, \Delta'')$. This implies $(\Delta, \Delta')\colon (\Delta'') = (\Delta, \Delta')$. Thus Δ'' is a non-zero divisor of $\mathbb{C}[T_{S\cup S'}]/(\Delta, \Delta')$. Because the factor ring $\mathbb{C}[T_{S\cup S'}, (\Delta'')^{-1}]/(\Delta, \Delta')$ is a domain, so is the factor ring $\mathbb{C}[T_{S\cup S'}]/(\Delta, \Delta')$. This shows (ii).

Case 2: Suppose $n + 1 \in S \cap S'$. Then this case reduces into Case 1.

Case 3: Suppose $n + 1 \in S \setminus S'$. Choose $S'' \in \left(\binom{[n+1]}{\ell+1}\right)$ such that $S \cap S' \subset S'' \subset S \cup S'$, $|S \cap S''| = \ell$, and $n + 1 \in S''$. The rest of the proof is exactly the same as Case 1. $\square$

Proof of Proposition 5. Since $\mathcal{A}$ is essential and not of general position, one has $\ell + 1 \leqslant n$.

(i) By Theorem 6, Δ_S is an irreducible polynomial. Thus C is irreducible and $\overline{\mathsf{B}} = \mathsf{C}$.

(ii) Let $n + 1 \notin S$. Let J be the ideal generated by the partial derivatives of Δ_S. Because of the Laplace expansion formula for $\det(\mathsf{T}_S)$, J is generated by the ℓ-minors of T_S. Thus any singular point $\boldsymbol{t}$ of B lies in $\mathsf{C}_{\{S'\}}$ for any $S' \in \left(\binom{[n+1]}{\ell+1}\right)$ with $|S \cap S'| = \ell$. Thus $\boldsymbol{t} \notin \mathsf{B}$. We can similarly prove the assertion when $n + 1 \in S$.

(iii) Let $S' \in \left(\binom{[n+1]}{\ell+1}\right) \setminus \{S\}$. Note $\mathsf{D}_{S'} = \mathsf{C}_{\{S,S'\}}$. If $|S \cap S'| \leqslant \ell - 1$, then $(\Delta_S, \Delta_{S'})$ is a prime ideal by Lemma 7(i). Thus $\mathsf{D}_{S'} = \mathsf{C}_{\{S,S'\}}$ is irreducible. If $|S \cap S'| = \ell$, then $(\Delta_S, \Delta_{S'}) = I_\ell(\mathsf{T}_{S\cap S'}) \cap I_{\ell+1}(\mathsf{T}_{S\cup S'})$ by Lemma 7(ii). If $\ell \geqslant 2$, this is a primary decomposition of $(\Delta_S, \Delta_{S'})$. Let $\{p\} = S \setminus S'$ and $\{q\} = S' \setminus S$. Then

$$\mathsf{D}_{S'} = \mathsf{C}_{\{S,S'\}} = \mathsf{C}_{\langle S-p\rangle} \cup \mathsf{C}_{\langle S+q\rangle}$$

is the decomposition of $\mathsf{D}_{S'}$ into irreducible components. The cardinality of the set $\{S' \in \left(\binom{[n+1]}{\ell+1}\right) \mid |S \cap S'| \leqslant \ell - 1\}$ is equal to $\binom{n+1}{\ell+1} - 1 - (n-\ell)(\ell+1)$. Thus the total number of irreducible components of $\mathsf{D} = \bigcup_{S' \in \mathcal{J}(\mathcal{A}_\infty)^c} \mathsf{D}_{S'}$ is equal to

$$\binom{n+1}{\ell+1} - 1 - (n-\ell)(\ell+1) + (\ell+1) + (n-\ell) = \binom{n+1}{\ell+1} - \ell(n-\ell-1).$$

If $\ell = 1 = |S \cap S'|$, then the ideal $I_\ell(\mathsf{T}_{S \cap S'})$ does not define a subvariety of $((\mathbb{CP}^\ell)^*)^n$. Thus $\mathsf{D}_{S'} = \mathsf{C}_{\langle S+q \rangle}$ where $\{q\} = S' \setminus S$. Therefore the total number of irreducible components of $\mathsf{D} = \bigcup_{S' \in \mathcal{J}(\mathcal{A}_\infty)^c} \mathsf{D}_{S'}$ is equal to

$$\binom{n+1}{2} - 1 - 2(n-1) + (n-1) = n(n-1)/2. \qquad \square$$

3. Logarithmic Gauss–Manin connections

Let $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$ be essential. We fix $\mathcal{A}$ in the rest of this section and write $\mathsf{B} = \mathsf{B}_{\mathcal{A}}$. Then, as we saw in the previous section, B may be considered a moduli space of the family of essential simple affine ℓ-arrangements which are combinatorially equivalent to $\mathcal{A}$. Let

$$\boldsymbol{t} = \left(\left(t_1^{(0)} : \cdots : t_1^{(\ell)}\right), \left(t_2^{(0)} : \cdots : t_2^{(\ell)}\right), \ldots, \left(t_n^{(0)} : \cdots : t_n^{(\ell)}\right)\right)$$

be the homogeneous coordinates for $((\mathbb{CP}^\ell)^*)^n$. Let $\boldsymbol{u} = (u_1, \ldots, u_\ell)$ be the standard coordinates for $\mathbb{C}^\ell$. Define

$$\mathsf{M} = \left\{(\boldsymbol{u}, \boldsymbol{t}) \in \mathbb{C}^\ell \times \left(\left(\mathbb{CP}^\ell\right)^*\right)^n \mid \boldsymbol{t} \in \mathsf{B},\ t_i^{(0)} + \sum_{j=1}^{\ell} t_i^{(j)} u_j \neq 0\ (i = 1, \ldots, n)\right\}.$$

Let

$$\pi : \mathsf{M} \to \mathsf{B}$$

be the projection defined by $\pi(\boldsymbol{u}, \boldsymbol{t}) = \boldsymbol{t}$. Then the fiber $\mathsf{M}_{\boldsymbol{t}} := \pi^{-1}(\boldsymbol{t})$ is the complement of the affine arrangement $\mathcal{A}_{\boldsymbol{t}}$ whose hyperplanes are defined by the equations $\alpha_i := t_i^{(0)} + \sum_{j=1}^{\ell} t_i^{(j)} u_j = 0$ $(i = 1, \ldots, n)$. Thus $\pi : \mathsf{M} \to \mathsf{B}$ is the complete family of essential simple affine arrangements in $\mathbb{C}^\ell$ which are combinatorially equivalent to $\mathcal{A}$. A result of Randell [7] implies that π is a fiber bundle over (the smooth part of) B.

We assume that d is the exterior differential operator with respect to the coordinates $\boldsymbol{u} = (u_1, \ldots, u_\ell)$ of $\mathbb{C}^\ell$ of the fiber and that $\omega_i := d \log \alpha_i = d\alpha_i / \alpha_i$ for $1 \leqslant i \leqslant n$ and

$$\omega_\lambda := \sum_{i=1}^{n} \lambda_i \omega_i, \qquad \nabla_\lambda : \Omega_M^p \to \Omega_M^{p+1}, \qquad \nabla_\lambda \eta := d\eta + \omega_\lambda \wedge \eta.$$

In this section we will compute covariant derivatives of differential forms in the fiber along the direction of the base.

Definition 8. Let d' be the exterior differential operator with respect to the homogeneous coordinates $\boldsymbol{t}$ of $((\mathbb{CP}^\ell)^*)^n$. For $1 \leqslant i \leqslant n$ define $\omega_i' := d' \log(\alpha_i / t_i^{(0)}) = (d'\alpha_i/\alpha_i) - (d't_i^{(0)}/t_i^{(0)})$ and

$$\omega_\lambda' := \sum_{i=1}^n \lambda_i \omega_i', \qquad \nabla_\lambda' : \Omega_M^p \to \Omega_M^{p+1}, \qquad \nabla_\lambda' \eta := d'\eta + \omega_\lambda' \wedge \eta.$$

Our next aim is to compute the operator ∇_λ' explicitly. For $S = (j_1, \ldots, j_m)$, $j_1 < \cdots < j_m$, write $S_k = (j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_m)$ $(1 \leqslant k \leqslant m)$ and $(S, j) = (j_1, \ldots, j_m, j)$ for $j \in [n+1] \setminus S$.

Definition 9. Let $T = (i_1, \ldots, i_\ell)$, with $i_k \in [n]$ $(1 \leqslant k \leqslant \ell)$. Write

$$\omega_T := \omega_{i_1} \wedge \cdots \wedge \omega_{i_\ell}, \qquad \zeta_T := \sum_{k=1}^{\ell} (-1)^{k+1} \omega_{i_k}' \wedge \omega_{T_k}.$$

The following computation was suggested by a method employed in [1].

Proposition 10.

$$\nabla_\lambda' \omega_T = -\nabla_\lambda \zeta_T + \sum_{j \in [n] \setminus T} \lambda_j \sum_{k=1}^{\ell+1} (-1)^{k+1} \omega_{(T,j)_k} \wedge d' \log\left(\frac{\Delta_{(T,j)}}{\Delta_{((T,j)_k, n+1)}} \right).$$

This proposition is an immediate consequence of the following two lemmas.

Lemma 11.

$$\nabla_\lambda' \omega_T + \nabla_\lambda \zeta_T = \sum_{j \in [n] \setminus T} \lambda_j \sum_{k=1}^{\ell+1} (-1)^{k+1} \omega_{(T,j)_k} \wedge \omega_{i_k}'.$$

Proof. Since d and d' operate in different variables, $dd' + d'd = 0$. This gives $d'\omega_T + d\zeta_T = 0$ used in the calculation below.

$$\begin{aligned}
&\nabla_\lambda' \omega_T + \nabla_\lambda \zeta_T \\
&\quad = d'\omega_T + \omega_\lambda' \wedge \omega_T + d\zeta_T + \omega_\lambda \wedge \zeta_T \\
&\quad = \sum_{k=1}^{\ell} (-1)^{k+1} \big(d'\omega_{i_k}\big) \wedge \omega_{T_k} + \omega_\lambda' \wedge \omega_T + \sum_{k=1}^{\ell} (-1)^{k+1} \big(d\omega_{i_k}'\big) \wedge \omega_{T_k} \\
&\qquad - \sum_{k=1}^{\ell} \lambda_{i_k} \omega_{i_k}' \wedge \omega_T + \sum_{j \in [n] \setminus T} \lambda_j \sum_{k=1}^{\ell} (-1)^{\ell+k+1} \omega_{i_k}' \wedge \omega_{(T_k, j)}
\end{aligned}$$

$$= \Bigg(\sum_{j\in[n]\setminus T} \lambda_j \omega'_j\Bigg) \wedge \omega_T + \sum_{j\in[n]\setminus T} \lambda_j \sum_{k=1}^{\ell} (-1)^{\ell+k+1} \omega'_{i_k} \wedge \omega_{(T_k,j)}$$

$$= \sum_{j\in[n]\setminus T} \lambda_j \sum_{k=1}^{\ell+1} (-1)^{k+1} \omega_{(T,j)_k} \wedge \omega'_{i_k}. \qquad \square$$

Lemma 12. *For $S \in \left(\binom{[n+1]}{\ell+1}\right)$, we have*

$$\sum_{k=1}^{\ell+1} (-1)^{k+1} \omega_{S_k} \wedge \omega'_{j_k} = \sum_{k=1}^{\ell+1} (-1)^{k+1} \omega_{S_k} \wedge d' \log\left(\frac{\Delta_S}{\Delta_{(S_k,n+1)}}\right).$$

Proof. Note that

$$\Delta_S = \sum_{k=1}^{\ell+1} (-1)^{k+1} t^{(0)}_{j_k} \Delta_{(S_k,n+1)} = \sum_{k=1}^{\ell+1} (-1)^{k+1} \alpha_{j_k} \Delta_{(S_k,n+1)}$$

by the Laplace expansion. Let

$$\alpha_S := \alpha_{j_1} \wedge \cdots \wedge \alpha_{j_{\ell+1}}, \qquad d\boldsymbol{u} := du_1 \wedge \cdots \wedge du_\ell.$$

We compute

$$\sum_{k=1}^{\ell+1} (-1)^{k+1} \omega_{S_k} \wedge \big(d' \log \Delta_S\big)$$

$$= \frac{1}{\alpha_S} \sum_{k=1}^{\ell+1} (-1)^{k+1} \alpha_{i_k} \Delta_{(S_k,n+1)} (d\boldsymbol{u}) \wedge \big(d' \log \Delta_S\big)$$

$$= \frac{1}{\alpha_S} \Delta_S (d\boldsymbol{u}) \wedge \big(d' \log \Delta_S\big) = \frac{1}{\alpha_S} (d\boldsymbol{u}) \wedge \big(d' \Delta_S\big)$$

$$= \frac{1}{\alpha_S} (d\boldsymbol{u}) \wedge d' \left(\sum_{k=1}^{\ell+1} (-1)^{k+1} \alpha_{j_k} \Delta_{(S_k,n+1)}\right)$$

$$= \frac{1}{\alpha_S} (d\boldsymbol{u}) \wedge \left[\sum_{k=1}^{\ell+1} (-1)^{k+1} \big\{\big(d'\alpha_{j_k}\big) \Delta_{(S_k,n+1)} + \alpha_{j_k} \big(d' \Delta_{(S_k,n+1)}\big)\big\}\right]$$

$$= \sum_{k=1}^{\ell+1} (-1)^{k+1} \big\{\omega_{S_k} \wedge \omega'_{j_k} + \omega_{S_k} \wedge \big(d' \log \Delta_{(S_k,n+1)}\big)\big\}.$$

This shows the lemma. $\square$

For $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\boldsymbol{t} \in \mathsf{B}$, recall the rank-one local system $\mathcal{L}_\lambda$ on $\mathsf{M}_{\boldsymbol{t}} = \pi^{-1}(\boldsymbol{t})$ so that $\mathcal{L}_\lambda$ has monodromy $\exp(-2\pi\sqrt{-1}\lambda_i)$ around $H_i = \{\alpha_i = 0\}$ $(i = 1, \ldots, n)$.

Theorem 13 [2,8]. *For a "generic" $\lambda \in \mathbb{C}^n$,*

(1) *$H^p(\mathsf{M}_{\boldsymbol{t}}, \mathcal{L}_{\boldsymbol{t}}) = 0$ $(p \neq \ell)$ and $\dim H^\ell(\mathsf{M}_{\boldsymbol{t}}, \mathcal{L}_{\boldsymbol{t}})$ is equal to $\beta = |\chi(M_{\boldsymbol{t}})|$, where χ stands for the Euler Poincaré characteristic,*

(2) *there exists a natural* (*twisted*) *de Rham isomorphism*

$$A^\ell/\omega_\lambda \wedge A^{\ell-1} \xrightarrow{\sim} H^\ell(\mathsf{M}_{\boldsymbol{t}}, \mathcal{L}_{\boldsymbol{t}}),$$

where $A^\cdot = \bigoplus_{q=0}^{\ell} A^q$ *is the Orlik–Solomon algebra* [5], [6, 3.45] *of* $\mathcal{A}_{\boldsymbol{t}}$. □

For explicit conditions for the "genericity", see [2,8]. In the rest of the paper, we assume that λ is generic in the sense of [8, 4.3 (Mon)**].

Since $\nabla_\lambda \circ \nabla'_\lambda + \nabla'_\lambda \circ \nabla_\lambda = 0$ and

$$H^\ell(\mathsf{M}_{\boldsymbol{t}}, \mathcal{L}_{\boldsymbol{t}}) \simeq A^\ell/\omega_\lambda \wedge A^{\ell-1} = A^\ell/\nabla_\lambda A^{\ell-1},$$

the operator ∇'_λ induces a $\mathbb{C}$-minear map

$$\nabla'_\lambda : H^\ell(\mathsf{M}_{\boldsymbol{t}}, \mathcal{L}_{\boldsymbol{t}}) \to H^\ell(\mathsf{M}_{\boldsymbol{t}}, \mathcal{L}_{\boldsymbol{t}}) \otimes \Omega^1(\log \mathsf{D})$$

by Proposition 10. Here $\Omega^1(\log \mathsf{D})$ is the space of meromorphic 1-forms on (the smooth part of) $\overline{\mathsf{B}}$ with logarithmic poles along $\mathsf{D} = \overline{\mathsf{B}} \setminus \mathsf{B}$. Let

$$\mathsf{D} = \bigcup_{s=1}^{t} \mathsf{D}_s$$

be the irreducible decomposition. For each irreducible component D_s and $S' \in \mathcal{J}(\mathcal{A}_\infty)^c$, define

$$\operatorname{mult}(S', \mathsf{D}_s) := \text{the order of zeros of } \ \Delta_{S'}|_{\overline{B}} \text{ along } \mathsf{D}_s$$

and

$$\begin{aligned} \Gamma(\mathsf{D}_s) :&= \{S' \in \mathcal{J}(\mathcal{A}_\infty)^c \mid \operatorname{mult}(S', \mathsf{D}_s) \geqslant 1\} \\ &= \{S' \in \mathcal{J}(\mathcal{A}_\infty)^c \mid \Delta_{S'}|_{\overline{B}} \text{ vanishes on } \mathsf{D}_s\}. \end{aligned}$$

We denote the logarithmic 1-form on $\overline{B}$ with simple logarithmic pole along D_s by $d' \log \mathsf{D}_s$ by abuse of notation. It can be locally expressed as $d \log f$ where $f = 0$ is a local defining equation for D_s. For $\omega \in A^\ell$, let $[\omega] \in H^\ell(\mathsf{M}_{\boldsymbol{t}}, \mathcal{L}_{\boldsymbol{t}})$ be its (twisted) de Rham cohomology class. Then, by Proposition 10, we immediately have

Theorem 14. *We have*

$$\nabla'_\lambda = \sum_{s=1}^{t} \nabla'_{\lambda,s} \otimes d' \log \mathsf{D}_s,$$

where $\nabla'_{\lambda,s} \in \operatorname{End}(H^\ell(\mathsf{M}_{\boldsymbol{t}}, \mathcal{L}_{\boldsymbol{t}}))$ *and, for* $T \in \left(\!\binom{[n+1]}{\ell+1}\!\right)$,

$$\begin{aligned} \nabla'_{\lambda,s}[\omega_T] = & \sum_{(T,j) \in \Gamma(\mathsf{D}_s)} \operatorname{mult}((T,j), \mathsf{D}_s) \lambda_j \sum_{k=1}^{\ell+1} (-1)^{k+1} [\omega_{(T,j)_k}] \\ & - \sum_{((T,j)_k, n+1) \in \Gamma(\mathsf{D}_s)} \operatorname{mult}(((T,j)_k, n+1), \mathsf{D}_s)(-1)^{k+1} \lambda_j [\omega_{(T,j)_k}]. \end{aligned}$$

Although Theorem 14 determines ∇'_λ and $\nabla'_{\lambda,s}$ completely, it is desirable to express each $\nabla'_{\lambda,s}$ explicitly in terms of a basis for $H^\ell(\mathsf{M}_t, \mathcal{L}_t)$. We propose to use the β**nbc** basis for this purpose. The β**nbc** basis is a combinatorially constructed basis for $H^\ell(\mathsf{M}_t, \mathcal{L}_t)$ in [3, 3.9]. When $\mathcal{A}$ is of general position, the set

$$\big\{[\eta_T] \in H^\ell(\mathsf{M}_t, \mathcal{L}_t) \mid T = (i_1, \dots, i_\ell),\ 2 \leqslant i_1 < \dots < i_\ell \leqslant n\big\}$$

is the β**nbc** basis, where

$$\eta_T := \lambda_{i_1} \cdots \lambda_{i_\ell} \omega_T.$$

In this case the expression of each $\nabla'_{\lambda,s}$ in terms of the β**nbc** basis is obtained in [1, Chapter 3 §8]. When B is one-codimensional in $((\mathbb{CP}^\ell)^*)^n$, the explicit formula is given in the next section. In general, it is not difficult to see from [3, 3.9] that $[\omega_T] \in H^\ell(\mathsf{M}_t, \mathcal{L}_t)$ is uniquely expresed as a linear combination of the β**nbc** basis $[\Xi_1], \dots, [\Xi_\beta] \in H^\ell(\mathsf{M}_t, \mathcal{L}_t)$ with coefficients lying in $\widetilde{\mathbb{Q}[\lambda]} := \mathbb{Q}[\lambda_1, \dots, \lambda_n, \{\lambda_X^{-1}\}]$, where $\lambda_X^{-1} = 1/(\sum_{X \subseteq H_j} \lambda_j)$ runs over the set $\{X \mid X \text{ is a dense edge}\}$. Recall that $\mathcal{H}_\ell$ is the rank β local system coming from the topological fibration $\pi : \mathsf{M} \to \mathsf{B}$. Then we have

Theorem 15. *The $\beta \times \beta$-matrix Ω, which satisfies the system of differential quations*

$$d' \begin{pmatrix} \int_\sigma \Phi_\lambda \Xi_1 \\ \cdots \\ \cdots \\ \int_\sigma \Phi_\lambda \Xi_\beta \end{pmatrix} = \Omega \wedge \begin{pmatrix} \int_\sigma \Phi_\lambda \Xi_1 \\ \cdots \\ \cdots \\ \int_\sigma \Phi_\lambda \Xi_\beta \end{pmatrix}$$

*for any (local) section σ of $\mathcal{H}_\ell$, the β**nbc** basis $[\Xi_1], \dots, [\Xi_\beta] \in H^\ell(\mathsf{M}_t, \mathcal{L}_t)$ and $\Phi_\lambda = \alpha_1^{\lambda_1} \cdots \alpha_n^{\lambda_n}$, has logarithmic poles along D with coefficients lying in $\widetilde{\mathbb{Q}[\lambda]}$.*

Proof. The integral $\int_\sigma \Phi_\lambda \Xi$ depends only upon the cohomology class $[\Xi] \in H^\ell(\mathsf{M}_t, \mathcal{L}_t)$. By Proposition 10, there exists a unique $\beta \times \beta$-matrix Ω such that

$$\begin{pmatrix} \nabla'_{\lambda,s}[\Xi_1] \\ \cdots \\ \cdots \\ \nabla'_{\lambda,s}[\Xi_\beta] \end{pmatrix} = \Omega \wedge \begin{pmatrix} [\Xi_1] \\ \cdots \\ \cdots \\ [\Xi_\beta] \end{pmatrix}.$$

Then Ω satisfies the desired properties. $\square$

Thus the connection $d' - \Omega\wedge$ on $\mathcal{O}_{\mathsf{B}}^\beta$ is a logarithmic Gauss–Manin connection and its flat sections are given by

$$\left\{ \begin{pmatrix} \int_\sigma \Phi_\lambda \Xi_1 \\ \cdots \\ \cdots \\ \int_\sigma \Phi_\lambda \Xi_\beta \end{pmatrix} \,\middle|\, \sigma \text{ is a local section of } \mathcal{H}^\ell \right\}.$$

4. The codimension one case

Suppose that the codimension of $\mathsf{B}_{\mathcal{A}}$ in $((\mathbb{CP}^\ell)^*)^n$ is one in this section. Then $\mathcal{J}(\mathcal{A}_\infty) = \{S\}$ for some $S \in \left(\binom{[n+1]}{\ell+1}\right)$. There are two cases: $n+1 \notin S$ (Case A) or $n+1 \in S$ (Case B). By permuting the hyperplanes if necessary, one can assume that $S = (1, 2, \ldots, \ell+1)$ (Case A) or $S = (n-\ell+1, n-\ell+2, \ldots, n+1)$ (Case B). It is easy to see that the β**nbc** basis for $H^\ell(\mathsf{M}_t, \mathcal{L}_t)$ is given by $\{[\eta_T] \in H^\ell(\mathsf{M}_t, \mathcal{L}_t) \mid T \in \beta\mathbf{nbc}\}$, where

$$\beta\mathbf{nbc} = \big\{(j_1, \ldots, j_\ell) \mid 2 \leqslant j_1 < \cdots < j_\ell \neq \ell+1\big\} \text{ (Case A)}$$

or

$$\beta\mathbf{nbc} = \big\{(j_1, \ldots, j_\ell) \mid 2 \leqslant j_1 < \cdots < j_\ell,\ j_1 \neq n-\ell+1\big\} \text{ (Case B).}$$

We will express $\nabla'_{\lambda,s}[\eta_T]$, $T \in \beta\mathbf{nbc}$, as a linear combination of $\{[\eta_{T'}] \in H^\ell(\mathsf{M}_t, \mathcal{L}_t) \mid T' \in \beta\mathbf{nbc}\}$ with coefficients in $\widetilde{\mathbb{Q}[\lambda]}$. (It will turn out that all the coefficients lie in $\sum_{i=1}^n \mathbb{Z}\lambda_i$.) In the following formulas for $\nabla'_{\lambda,s}[\eta_T]$ we use the notation

$$\varepsilon\big(T, T'\big) = (-1)^{p+q}$$

if $T, T' \subseteq [n]$, $|T| = |T'| = \ell$, $|T \cap T'| = \ell - 1$, $U = T \cup T'$, $T = U_p$ and $T' = U_q$. Define $\varepsilon(T, T') = 1$ if $T = T'$. For example, $\varepsilon(23, 35) = 1$ because $U = 235$, $T = 23 = U_3$, $T' = 35 = U_1$.

Case A. Let $S = (1, 2, \ldots, \ell+1)$.

Type A.I. Let $\mathsf{D}_s = \mathsf{C}_{\{S,S'\}}$ for $S' \in \left(\binom{[n+1]}{\ell+1}\right)$ with $|S \cap S'| \leqslant \ell - 1$ (Proposition 5(iii)). In this case, $\Gamma(\mathsf{D}_s) = \{S, S'\}$ and $\operatorname{mult}(S', \mathsf{D}_s) = 1$ because the ideal $(\Delta_S, \Delta_{S'})$ is prime by Lemma 7(ii).

Case A.I.1. Suppose $S' \cap \{1, n+1\} = \emptyset$.

- If $S' \supset T \in \beta\mathbf{nbc}$, then

$$\nabla'_{\lambda,s}[\eta_T] = \sum_{k=1}^{\ell+1} \varepsilon\big(T, S'_k\big)\lambda_{S' \setminus S'_k}[\eta_{S'_k}],$$

  where $S'_k = (i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_{\ell+1})$ and $\lambda_{S' \setminus S'_k} = \lambda_{i_k}$ if $S' = (i_1, \ldots, i_{\ell+1})$.
- Otherwise, $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case A.I.2. Suppose $S' \cap \{1, n+1\} = \{n+1\}$.

- If $S' \supset T \in \beta\mathbf{nbc}$, then $T = S'_{\ell+1} = S' \setminus \{n+1\}$ and

$$\nabla'_{\lambda,s}[\eta_T] = -\bigg(\sum_{j \in [n] \setminus T} \lambda_j\bigg)[\eta_T].$$

- If $T \in \beta\mathbf{nbc}$ with $|T \cap S'| = \ell - 1$, then

$$\nabla'_{\lambda,s}[\eta_T] = -\varepsilon\big(T, S'_{\ell+1}\big)\lambda_{T \setminus S'}[\eta_{S'_{\ell+1}}].$$

- Otherwise, $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case A.I.3. Suppose $S' \cap \{1, n+1\} = \{1\}$.
- If $S' \supset T \in \beta\mathbf{nbc}$, then $T = S'_1 = S' \setminus \{1\}$ and

$$\nabla'_{\lambda,s}[\eta_T] = \Bigl(\sum_{j \in S'} \lambda_j\Bigr)[\eta_T] - \sum_{\substack{T' \in \beta\mathbf{nbc} \\ |T \cap T'| = \ell - 1}} \varepsilon\bigl(T, T'\bigr)\lambda_{T \setminus T'}[\eta_{T'}].$$

- Otherwise, $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case A.I.4. Suppose $S' \cap \{1, n+1\} = \{1, n+1\}$.
- If $T \in \beta\mathbf{nbc}$ with $|T \cap S'| = \ell - 1$, then $S' \setminus \{1, n+1\} \subset T$ and

$$\nabla'_{\lambda,s}[\eta_T] = -\lambda_{T \setminus S'} \sum_{\substack{T' \in \beta\mathbf{nbc} \\ T' \supset T \cap S'}} [\eta_{T'}].$$

- Otherwise, $\nabla'_{\lambda,s}[\eta_T] = 0$.

Type A.II. Suppose $\ell \geqslant 2$. Let $\mathsf{D}_s = \mathsf{C}_{\langle S-p \rangle}$ where $p \in S = (1, 2, \ldots, \ell + 1)$, $S - p :=$ $S \setminus \{p\}$, and $\langle S - p \rangle = \{S' \in \bigl(\binom{[n+1]}{\ell+1}\bigr) \mid S' \supset S - p\}$. In this case, $\Gamma(\mathsf{D}_s) = \langle S - p \rangle$ and $\mathrm{mult}(S', \mathsf{D}_s) = 1$ for $S' \in \langle S - p \rangle$, $S' \neq S$, because the ideal $(\Delta_S, \Delta_{S'})$ is a radical ideal by Lemma 7(i).

Case A.II.1. Suppose $p \neq 1$.
- If $T \in \beta\mathbf{nbc}$ with $|T \cap (S - p)| = \ell - 1$, then

$$\nabla'_{\lambda,s}[\eta_T] = \Bigl(\sum_{j \in S-p} \lambda_j\Bigr)[\eta_T] - \sum_{\substack{T' \in \beta\mathbf{nbc} \\ |T \cap T'| = \ell - 1 \\ |T' \cap (S-p)| = \ell - 2}} \varepsilon\bigl(T, T'\bigr)\lambda_{T \setminus T'}[\eta_{T'}].$$

- Otherwise, $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case A.II.2. Suppose $p = 1$.
- If $T \in \beta\mathbf{nbc}$ with $|T \cap (S - 1)| = \ell - 1$, then

$$\nabla'_{\lambda,s}[\eta_T] = \lambda_{(S-1) \setminus T}[\eta_T] + \sum_{\substack{T' \in \beta\mathbf{nbc} \\ |T \cap T'| = \ell - 1 \\ T' \subset S \cup T}} \varepsilon\bigl(T, T'\bigr)\lambda_{T \setminus T'}[\eta_{T'}].$$

- Otherwise, $\nabla'_{\lambda,s}[\eta_T] = 0$.

Type A.III. Let $\mathsf{D}_s = \mathsf{C}_{\langle S+q \rangle}$ where $q \in [n+1] \setminus S = (\ell+2, \ell+3, \ldots, n+1)$, and $S + q :=$ $S \cup \{q\}$, and $\langle S + q \rangle = \{S' \in \bigl(\binom{[n+1]}{\ell+1}\bigr) \mid S' \subset S + q\}$. In this case, $\Gamma(\mathsf{D}_s) = \langle S + q \rangle$ and $\mathrm{mult}(S', \mathsf{D}_s) = 1$ for $S' \in \langle S + q \rangle$, $S' \neq S$, because the ideal $(\Delta_S, \Delta_{S'})$ is a radical ideal by Lemma 7(i).

Case A.III.1. Suppose $q \neq n+1$.

- If $T \in \beta\mathbf{nbc}$ with $T \subset S+q$, then

$$\nabla'_{\lambda,s}[\eta_T] = \left(\sum_{j\in S+q}\lambda_j\right)[\eta_T] - \sum_{\substack{T'\in\beta\mathbf{nbc}\\ |T\cap T'|=\ell-1\\ |T'\cap(S+q)|=\ell-1}} \varepsilon\big(T,T'\big)\lambda_{T\setminus T'}[\eta_{T'}].$$

- Otherwise, $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case A.III.2. Suppose $q = n+1$.

- If $T \in \beta\mathbf{nbc}$ with $|T \cap S| = \ell - 1$, then

$$\nabla'_{\lambda,s}[\eta_T] = -\lambda_{T\setminus S}\sum_{\substack{T'\in\beta\mathbf{nbc}\\ T'\cap S=T\cap S}}[\eta_{T'}].$$

- Otherwise, $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case B. Let $S = (n-\ell+1,\dots,n+1)$.

Type B.I. Let $\mathsf{D}_s = \mathsf{C}_{\{S,S'\}}$ for $S' \in \left(\!\binom{[n+1]}{\ell+1}\!\right)$ with $|S\cap S'| \leqslant \ell-1$. For this type, we have the exact same formulas as Case A.

Type B.II. Suppose $\ell \geqslant 2$. Let $\mathsf{D}_s = \mathsf{C}_{\langle S-p\rangle}$.

Case B.II.1. Suppose $p \neq n+1$.

- If $T \in \beta\mathbf{nbc}$ with $|T\cap(S-p)| = \ell-1$, then

$$\nabla'_{\lambda,s}[\eta_T] = -\left(\sum_{j\notin S-p}\lambda_j\right)[\eta_T].$$

- If $T \in \beta\mathbf{nbc}$ with $|T\cap(S-p)| = \ell-2$, then

$$\nabla'_{\lambda,s}[\eta_T] = -\sum_{\substack{T'\in\beta\mathbf{nbc}\\ |T\cap T'|=\ell-1\\ |T'\cap(S-p)|=\ell-1}} \varepsilon\big(T,T'\big)\lambda_{T\setminus T'}[\eta_{T'}].$$

- Otherwise, $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case B.II.2. Same formulas as Case A.II.2.

Type B.III. Let $\mathsf{D}_s = \mathsf{C}_{\langle S+q\rangle}$ where $q \in [n+1]\setminus S = (\ell+2,\ell+3,\dots,n+1)$.

Case B.III.1. Suppose $q \neq 1$.

- If $T \in \beta\mathbf{nbc}$ with $T \subset S+q$, then

$$\nabla'_{\lambda,s}[\eta_T] = -\left(\sum_{j\notin S+q}\lambda_j\right)[\eta_T].$$

- If $T \in \beta\mathbf{nbc}$ with $|T \cap (S+q)| = \ell - 1$, then

$$\nabla'_{\lambda,s}[\eta_T] = -\sum_{\substack{T' \in \beta\mathbf{nbc} \\ |T \cap T'| = \ell - 1 \\ T' \subset (S+q)}} \varepsilon\big(T, T'\big)\lambda_{T \setminus T'}[\eta_{T'}].$$

- Otherwise, $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case B.III.2. Same formulas as Case A.III.2.

Summarizing Cases A and B above, we have

Theorem 16. *Suppose that* $\mathsf{B} = \mathsf{B}_{\mathcal{A}}$ *is one-codimensional in* $((\mathbb{CP}^\ell)^*)^n$. *Let* $\mathcal{J}(\mathcal{A}_\infty) = \{S\}$, $\mathsf{D} = \overline{\mathsf{B}} \setminus \mathsf{B}$ *and* $\mathsf{D} = \bigcup_{s=1}^t \mathsf{D}_s$ *be the irreducible decomposition. Then*

(1) *the logarithmic Gauss–Manin connection matrix* Ω *in Theorem* 15 *can be expressed as* $\Omega = \sum_{s=1}^t \Omega_s \otimes d' \log \mathsf{D}_s$ *such that each* Ω_s *has its entries in* $\sum_{i=1}^n \mathbb{Z}\lambda_i$.

(2) *The eigenvalues of* Ω_s *are*:

(i) $\sum_{j \in S} \lambda_j$ *with multiplicity one and the rest are zero* (*if* D_s *is of Type* I *in Proposition* 5),

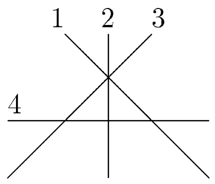
(ii) $\sum_{j \in S-p} \lambda_j$ *with multiplicity* $n - \ell - 1$ *and the rest are zero* (*if* D_s *is of Type* II), *or*

(iii) $\sum_{j \in S+q} \lambda_j$ *with multiplicity* ℓ *and the rest are zero* (*if* D_s *is of Type* III), *where we define* $\lambda_{n+1} := -\lambda_1 - \lambda_2 - \cdots - \lambda_n$.

(*The explicit formulas for* Ω_s *are given above when* $S = (1, 2, \ldots, \ell + 1)$ (*Case* A) *or* $S = (n - \ell + 1, n - \ell + 2, \ldots, n + 1)$ (*Case* B).)

Proof. Although the $\beta\mathbf{nbc}$ basis depends on the linear order on $\mathcal{A}$. It is known [3, 3.11] that two $\beta\mathbf{nbc}$ bases are connected by an integral unimodular matrix (without λ). Thus one can assume that $S = (1, 2, \ldots, \ell + 1)$ (when $n + 1 \notin S$) or $S = (n - \ell + 1, n - \ell + 2, \ldots, n + 1)$ (when $n + 1 \in S$). Use the above-mentioned explicit formulas for Cases A and B. □

Example 17. Let $\ell = 2$, $n = 4$, $S = (1, 2, 3)$ and $\mathcal{J}(\mathcal{A}_\infty) = \{S\}$.



$\mathcal{A}$

Write 123 for $(1, 2, 3)$ etc. The boundary divisor $\mathsf{D} = \overline{\mathsf{B}}_{\mathcal{A}} \setminus \mathsf{B}_{\mathcal{A}}$ has eight irreducible components that are given in Fig. 1.

The matrices Ω_s $(s = 1, \ldots, 8)$ in terms of the $\beta\mathbf{nbc}$ basis $\{[\eta_{24}], [\eta_{34}]\}$, are

$$\Omega_1 = \begin{pmatrix} -\lambda_2 & -\lambda_2 \\ -\lambda_3 & -\lambda_3 \end{pmatrix}, \qquad \Omega_2 = \begin{pmatrix} -\lambda_1 - \lambda_3 & 0 \\ \lambda_3 & 0 \end{pmatrix},$$

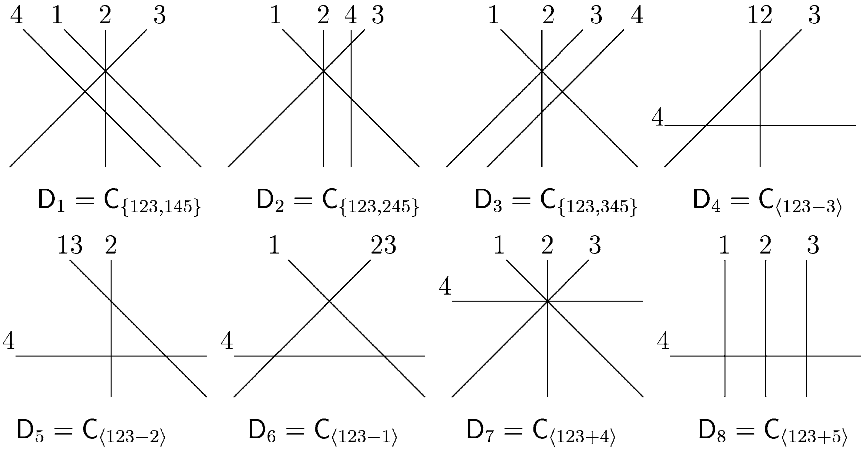


Fig. 1. Eight irreducible components.

$$\Omega_3 = \begin{pmatrix} 0 & \lambda_2 \\ 0 & -\lambda_1 - \lambda_2 \end{pmatrix}, \qquad \Omega_4 = \begin{pmatrix} \lambda_1 + \lambda_2 & \lambda_2 \\ 0 & 0 \end{pmatrix},$$

$$\Omega_5 = \begin{pmatrix} 0 & 0 \\ \lambda_3 & \lambda_1 + \lambda_3 \end{pmatrix}, \qquad \Omega_6 = \begin{pmatrix} \lambda_3 & -\lambda_2 \\ -\lambda_3 & \lambda_2 \end{pmatrix},$$

$$\Omega_7 = \begin{pmatrix} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 & 0 \\ 0 & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \end{pmatrix},$$

$$\Omega_8 = \begin{pmatrix} -\lambda_4 & 0 \\ 0 & -\lambda_4 \end{pmatrix}.$$

For an arbitrary arrangement $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$ and $\mathsf{B} = \mathsf{B}_{\mathcal{A}}$, it seems to be difficult to find explicit matrix presentations for ∇'_λ. Based upon our result for the codimension one case, it might be natural to ask the following questions:

Question 1. Does each entry of the matrix Ω_s lie in $\sum_{i=1}^n \mathbb{Z}\lambda_i$?

Question 2. Is B smooth?

Acknowledgement

The author would like to thank K. Kurano for providing proof of Lemma 7. The author would also like to thank K. Aomoto, J. Kaneko, P. Orlik and T. Terasoma for stimutating conversations on the subject.

References

[1] K. Aomoto, M. Kita, Hypergeometric Functions, Springer-Verlag, Berlin, 1994 (in Japanese).
[2] H. Esnault, V. Schechtman, E. Viehweg, Cohomology of local systems of the complement of hyperplanes, Invent. Math. 109 (1992) 557–561, Erratum, Invent. Math. 112 (1993) 447.
[3] M. Falk, H. Terao, βNBC-bases for cohomology of local systems on hyperplane complements, Trans. Amer. Math. Soc. 349 (1997) 189–202.
[4] M. Hochster, J.A. Eagon, Cohen–Macaulay rings, invariant theory, and the generic perfection of determinantal loci, Amer. J. Math. 93 (1971) 1020–1058.
[5] P. Orlik, L. Solomon, Combinatorics and topology of complements of hyperplanes, Invent. Math. 56 (1980) 167–189.
[6] P. Orlik, H. Terao, Arrangements of Hyperplanes, Springer-Verlag, Berlin, 1992.
[7] R. Randell, Lattice-isotopic arrangements are topologically isomorphic, Proc. Amer. Math. Soc. 107 (1989) 555–559.
[8] V. Schechtman, H. Terao, A. Varchenko, Local systems over complements of hyperplanes and the Kac–Kazhdan conditions for singular vectors, J. Pure Appl. Algebra 100 (1995) 93–102.